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Space–time noncommutativity and the (1 + 1) Higgs model

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Abstract

We compare the classical scattering of kinks in the (1 + 1) Higgs model with its analogous noncommutative counterpart. While at a classical level we are able to solve the scattering at all orders finding a smooth solution, at a noncommutative level we present only perturbative results, suggesting the existence of a smooth solution also in this case.

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1. Introduction

Recently, noncommutative gauge theories have been investigated in view of their eventual role for a better understanding of our physical world. The general approach is introducing noncommutativity as a one-parameter modification of gauge theories, but most of these studies do not give emphasis to a possible clash between noncommutativity and the basic principles on which particle physics is built.

In this sense, space–time noncommutativity has represented an exception to this approach, since this new type of nonlocality in time may be very dangerous to the maintenance of the basic rules of standard quantum field theory [1]. In fact, quantum field theories with space–time noncommutativity have no straightforward Hamiltonian quantization, which is usually the warranty for unitarity and causality.

There are two main positions regarding such theories: someone claims that these theories are physically inconsistent because they break perturbative unitarity of the S-matrix [1–4], someone else claims that restoring unitarity is possible modifying the definition of perturbation theory on noncommutative space–times [5–8].

In this paper, we attempt to give a contribution to this subject at a classical level comparing the scattering of wave packets in the commutative theory with the noncommutative case. Usually already in the classical setting it is possible to find the seeds of the obstacles in the quantization procedure. We have searched for a model which has some chance to be a solvable system and where nonperturbative effects can be detected.

We have chosen the Higgs model in (1 + 1)-dim. which allows for soliton solutions, called kinks, which interpolate between the possible vacua of the Higgs potential. The scattering of kinks is the natural setting in which to compare the commutative results with the noncommutative ones (see also [9]). To simplify the discussion, we have stretched the kinks such that the wavefronts are sharp (ultra-relativistic limit). At a commutative level we find that this ansatz is self-consistent at all orders of perturbation theory, and we are able to reduce the equations of motion to a nonlinear differential equation in a single variable that, due to the nonlinearity, we have studied numerically.

At a noncommutative level, the same ansatz is no longer stable and it is not possible to reduce the equations of motion to a single variable. We are obliged to analyse the problem perturbatively, also if the structure of the solution is still simple enough, that could enable us to reach a nonperturbative result with a deeper study.

We essentially show how noncommutativity acts on the scattering of kinks; the sharpness of wavefronts can be maintained only at a nonperturbative level in the commutative case, while a correction of the profiles of the wavefronts, which lose their sharpness, of the order of the noncommutative parameter $\sqrt{\theta}$ is needed in the noncommutative one. Let us note that this effect is present only when the two shock waves scatter each other and is not if the two waves proceed in the same direction. Of course the scattering is anticipated with respect to the commutative case due to this 'apparent width' of the wave packets. This implies that noncommutative kinks cannot be stretched to sharp wavefronts during the scattering process, a limitation not present in the classical case. Of course, it would be more interesting to interpret a complete noncommutative scattering solution, which we leave to a future research.

2. Higgs model for real scalar field

Our aim was to find an exactly solvable scattering problem to compare with the noncommutative case, possibly at a nonperturbative level. Our choice has been to consider the classical (1 + 1) Higgs model with a real scalar field defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\partial_\mu\phi + \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4}\lambda\phi^4. \tag{2.1}$$

The Higgs mechanics is based on making perturbation theory around the nontrivial minimum of the potential. To find it we need to introduce the corresponding Hamiltonian:

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi \qquad \mathcal{H} = \frac{1}{2}(\pi^2 + |\partial_i\phi|^2) + V(\phi) \qquad V(\phi) = \frac{1}{4}\lambda\phi^4 - \frac{1}{2}m^2\phi^2. \tag{2.2}$$

The minimum of the potential (2.2) is reached when

$$\frac{\partial V(\phi)}{\partial\phi} = 0 \rightarrow \phi^2 = \frac{m^2}{\lambda}. \tag{2.3}$$

Let us choose for simplicity, the coupling constants such that $m = \sqrt{\lambda}$ and $\phi = \pm 1$ are the two nontrivial minima.

The corresponding equations of motion

$$\Box\phi = \lambda\phi(1-\phi^2) \tag{2.4}$$

can be simplified by defining the light-cone variables:

$$u = \frac{x-t}{2} \qquad v = \frac{x+t}{2} \qquad \Box = -\partial_u\partial_v \qquad \partial_u\partial_v\phi = -\lambda\phi(1-\phi^2). \tag{2.5}$$

Analysing such equations one note that $\phi = 0, \pm 1$ are possible solutions. Let us generalize them by introducing as an ansatz a sum of step functions which interpolate between the three values $0, \pm 1$:

$$\phi^{(0)} = -1 + p(\Theta(u) - \Theta(v)) \qquad p = 1, 2. \tag{2.6}$$

Due to the dependence from single light-cone variables, $\phi^{(0)}$ is certainly a zero of the first member of equation (2.5); however the second member is null only if $t < 0$. For $t > 0$ the ansatz $\phi^{(0)}$ is no longer valid and must be generalized to

$$\phi = -1 + p(\Theta(u) - \Theta(v)) + f(uv)\Theta(v)\Theta(-u) \tag{2.7}$$

where the unknown function f is dependent on the single variable uv.[1] This interacting solution must evolve such that asymptotically, at infinite time and finite space, i.e., for $uv \to -\infty$, the function reaches a constant value. The particular value $f(-\infty)$ is determined by solving the equations of motion (2.5).

The solution with $p = 1$ has as 'in state' $\phi_{\text{in}} = 0$ (for $t \to -\infty$), while the 'out state' (for $t \to +\infty$) has to be determined by solving the equations of motion (2.5), and it is parametrized as $f_{p=1}(-\infty) - 2$.

The solution with $p = 2$ has as 'in state' $\phi_{\text{in}} = 1$ (for $t \to -\infty$) and the 'out state' $f_{p=2}(-\infty) - 3$.

In synthesis,

$$\begin{aligned} &p = 1 \qquad \phi_{\text{in}} = 0 \quad \to \quad \phi_{\text{out}} = f_{p=1}(-\infty) - 2 \\ &p = 2 \qquad \phi_{\text{in}} = 1 \quad \to \quad \phi_{\text{out}} = f_{p=2}(-\infty) - 3. \end{aligned} \tag{2.8}$$

By introducing the complete ansatz (2.7) into the equations of motion (2.5) one is able to show that the ansatz closes if all the following conditions are met:

$$\begin{aligned} &f(0) = 0 \qquad f'(uv) \quad \text{regular around } uv \sim 0 \\ &(uv)f''(uv) + f'(uv) = \lambda(f(uv) - p)(f(uv) - p - 1)(f(uv) - p - 2) \\ &p(p-1)(p-2) = 0. \end{aligned} \tag{2.9}$$

The ansatz closes for the values $p = 0, 1, 2$; since the value $p = 0$ is trivial, in the following we will discuss only the solution to the differential equation (2.9) for the values $p = 1, 2$.

By introducing the variable

$$x = -6\lambda uv \tag{2.10}$$

we are led to discuss the solution to the following equation:

$$xf''(x) + f'(x) + (f(x) - p)(f(x) - p - 1)(f(x) - p - 2)/6 = 0 \tag{2.11}$$

in the range $0 < x < +\infty$.

We will show that the nonlinear equation (2.11) is consistent with the following boundary values:

$$f_{p=1}(x = +\infty) = 1 \qquad f_{p=2}(x = +\infty) = 4 \tag{2.12}$$

leading to classify the possible scenarios:

$$\begin{aligned} &p = 1 \qquad \phi_{\text{in}} = 0 \quad \to \quad \phi_{\text{out}} = -1 \\ &p = 2 \qquad \phi_{\text{in}} = 1 \quad \to \quad \phi_{\text{out}} = 1. \end{aligned} \tag{2.13}$$

[1] To avoid confusion, we define the step functions as $\Theta(u) = 1$ if $u > 0$, or 0 otherwise, $\Theta(v) = 1$ if $v > 0$, or 0 otherwise, and $\Theta(-u) \equiv 1 - \Theta(u)$.

In the first case, the scattering of kinks allows us to describe the decay from the unstable state $\phi_{\mathrm{in}} = 0$ to the stable minimum of the potential $\phi_{\mathrm{out}} = -1$; in the second case, the scattering of kinks does not alter the stability of the minimum $\phi_{\mathrm{in}} = \phi_{\mathrm{out}} = 1$.

Let us start to solve (2.11) with $p = 1$. In this case, we define

$$\begin{aligned} &f(x) = 1 - g(x) \qquad g(0) = 1 \\ &xg''(x) + g'(x) + g(x)(g(x)+1)(g(x)+2)/6 = 0. \end{aligned} \tag{2.14}$$

Being a second-order differential equation, it seems that the only boundary value $g(0) = 1$ is not enough to determine completely the solution, but it turns out that another physical requirement is necessary to obtain a smooth solution, i.e., the absence of logarithmic terms around $x = 0$. The request is sufficient to determine the asymptotic value $g(+\infty) = 0$, which can be achieved or by a direct numerical computation with Mathematica, or with a careful inspection of the differential equation. We have done both checks and they completely agree.

Firstly, let us suppose that asymptotically $g(x) \to 0$ for $x \to +\infty$, then the nonlinearity can be avoided and the nonlinear problem (2.11) can be linearized to

$$\begin{aligned} &xg''(x) + g'(x) + g(x)/3 = 0 \\ &g(0) = 1 \qquad g(x) \text{ regular around } x = 0 \end{aligned} \tag{2.15}$$

which can be solved by the Bessel function:

$$g(x) = J_0\left(2\sqrt{\frac{x}{3}}\right). \tag{2.16}$$

Since it is well known that the asymptotic behaviour of the Bessel function

$$\lim_{x\to+\infty} J_0(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4}\right) \tag{2.17}$$

we conclude that the asymptotic value $g(+\infty) = 0$ is consistent with the hypothesis, and we have learned that the solution of (2.11) has asymptotic damped oscillations behaving as $x^{-\frac{1}{4}}$.

The difference between linear and nonlinear solutions is concentrated around $x = 0$; however, supposing that the value of $g(x)$ is always greater than -1, at the minima where $g'(x) = 0$, the value of the second derivative $g''(x)$ is always opposite to the value of the function $g(x)$, confirming the oscillating character of the solution around the value $g = 0$, also at the nonlinear level. At this point, we are ready to compare these considerations with a direct numerical computation with Mathematica.

We chose the following method. Firstly, we developed in power series the solution around zero and put the recurrence relations for the coefficients of the series into Mathematica. The power series truncated at, let us say, 100 steps has a certain convergence radius; inside it we can trust the approximated values of the function $g(x)$. Then, we used these approximated values to build the recurrence relations around another fixed point inside the convergence radius of the first power series, and we iterated the procedure. In this way, we have found the locus of the first six minima (these results should be taken as indicative values due to the imprecision of the extrapolated values)

$$\begin{aligned} &x = 11 \qquad &&g(x) = -0.62 \\ &x = 90 \qquad &&g(x) = -0.49 \\ &x = 244 \qquad &&g(x) = -0.36 \\ &x = 462 \qquad &&g(x) = -0.32 \\ &x = 745 \qquad &&g(x) = -0.28 \\ &x = 1089 \qquad &&g(x) = -0.25 \end{aligned} \tag{2.18}$$

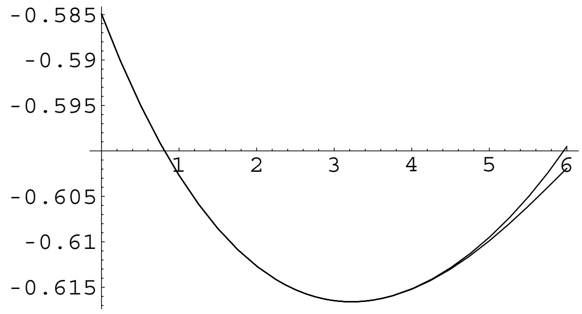

Figure 1. The first minimum for $p = 1$.

and of the first five local maxima:

$$\begin{aligned}
&x = 41 && g(x) = 0.37\\
&x = 161 && g(x) = 0.29\\
&x = 346 && g(x) = 0.25\\
&x = 598 && g(x) = 0.23\\
&x = 911 && g(x) = 0.21.
\end{aligned} \tag{2.19}$$

In figure 1 we have plotted with Mathematica two curves representing the approximated value of the function $g(x)$ calculated from the power series truncated at an even and odd number, respectively (for example, 100 and 99 steps).

By analysing these results we find agreement with all the preliminary discussion, since the minimum value of $g(x)$ (-0.62) is greater than -1, from which damped oscillations follow until reaching the linear behaviour (2.17). Thus, by combining numerical and analytic methods we have full control of the nonlinear equation (2.14).

Let us now discuss the solution to the equations of motion for $p = 2$, in which case the final state of the Higgs field $\phi_{\text{out}} = f_{p=2}(-\infty) - 3$. By defining $f(x) = 2(1 - g(x))$ and rescaling $x \to 2x$, we obtain the following differential equation for $g(x)$:

$$xg''(x) + g'(x) + g(x)(1 + g(x))(1 + 2g(x))/6 = 0. \tag{2.20}$$

At a first sight this equation looks very similar to the one discussed before, but in reality its solution is quite different. Firstly, we note that it is not clear what is the final point of oscillation. There are at least two possible choices:

(i) $g(x)$ oscillates around the value $g = 0$; then to be self-consistent, at the stationary points, the value of the second derivative must be opposite to the value of the function and this happens if the function is confined over the minimum value $-1/2$.
(ii) $g(x)$ oscillates around the value $g = -1$; this is possible if the local maxima and minima are confined under the maximum value $-1/2$.

Only with a numerical computation have we been able to discern the right value. By using the same method illustrated before we have found the locus of the first four minima

$$\begin{aligned}
&x = 41 && g(x) = -1.14\\
&x = 212 && g(x) = -1.10\\
&x = 506 && g(x) = -1.08\\
&x = 922 && g(x) = -1.07
\end{aligned} \tag{2.21}$$

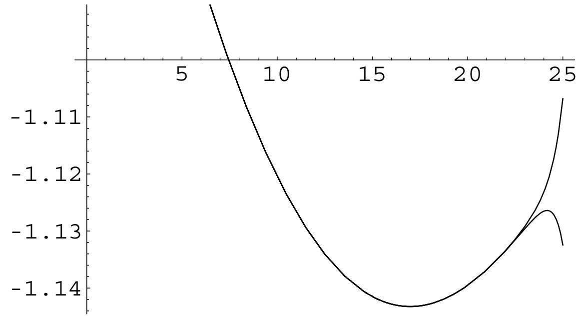

Figure 2. The first minimum for $p = 2$.

and of the first four local maxima:

$$\begin{aligned} x &= 108 \qquad & g(x) &= -0.85 \\ x &= 342 \qquad & g(x) &= -0.89 \\ x &= 697 \qquad & g(x) &= -0.91 \\ x &= 1173 \qquad & g(x) &= -0.92. \end{aligned} \tag{2.22}$$

We conclude that the possibility (ii) is realized (see figure 2), and therefore $g(x)$ has an asymptotic value -1, with oscillations that are damped by a factor $x^{-1/4}$, the typical factor of the Bessel function (2.17). This completes the demonstration of the boundary values depicted in (2.12).

3. Noncommutative case

We are going to deform the Higgs model with a noncommutative relation between the coordinates, for example,

$$[u, v] = \mathrm{i}\theta \quad \leftrightarrow \quad [x, t] = 2\mathrm{i}\theta. \tag{3.1}$$

This can be accomplished by deforming the ordinary product of fields into an associative star product as follows:

$$\phi_1(u, v) * \phi_2(u, v) = \lim_{u_1 \to u_2} \lim_{v_1 \to v_2} \mathrm{e}^{\mathrm{i}\frac{\theta}{2}(\partial_{u_1}\partial_{v_2} - \partial_{u_2}\partial_{v_1})} \phi_1(u_1, v_1)\phi_2(u_2, v_2). \tag{3.2}$$

The Lagrangian of the Higgs field with a noncommutative star product is defined as

$$\mathcal{L} = \int \mathrm{d}^2x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi * \phi * \phi * \phi \right) \tag{3.3}$$

and the equations of motion (with $m^2/\lambda = 1$) are nonlocal in time:

$$\partial_u\partial_v\phi = \lambda\phi * (\phi * \phi - 1). \tag{3.4}$$

It is not clear at this point how to complete the ansatz (2.6)

$$\phi^{(0)} = -1 + p(\Theta(u) - \Theta(v)) \tag{3.5}$$

in order to reduce the equations of motion (3.4) into a self-consistent set of equations. We prefer to set up a perturbative method. In this framework, the natural perturbative parameter is the coupling constant λ, and we develop the solution ϕ as a sum

$$\phi = \phi^{(0)} + \lambda\phi^{(1)} + O(\lambda^2). \tag{3.6}$$

Starting from the ansatz $\phi^{(0)}$ (3.5), we calculate the source (3.4) at the first perturbative order in λ. In the case of the ordinary Higgs model, we would obtain

$$\begin{aligned}\partial_u\partial_v\phi^{(1)} &= \lambda\phi^{(0)}(\phi^{(0)2}-1)\\ &= \lambda p(p-1)(p-2)(\Theta(u)-\Theta(v)) - 6\lambda p^2\Theta(-u)\Theta(v).\end{aligned} \tag{3.7}$$

For the special cases $p = 1, 2$ the first term in the second member cancels out and the main contribution comes from the term $\Theta(-u)\Theta(v)$ which is different from zero only for $t > 0$, and in the interval $-t < x < t$.

The interacting field $\phi^{(1)}$ is then proportional to $x^2 - t^2$

$$\phi^{(1)} = -6\lambda p^2 uv\Theta(-u)\Theta(v). \tag{3.8}$$

In the noncommutative case, we limit ourselves to a calculation of $\phi^{(1)}$ always starting from the ansatz $\phi^{(0)}$ but replacing the ordinary product with the star product

$$\partial_u\partial_v\phi^{(1)}_{\mathrm{NC}} = \lambda\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1). \tag{3.9}$$

However, we encounter the first difficulty, i.e., the star product involving step functions seems to be ill-defined, being a sum of infinite distributions. We will do the following trick, i.e., solving the star product using the Fourier transform.

As an exercise, let us calculate the ordinary product $\phi^{(0)2}$ using the Fourier transform of $\phi^{(0)}$. It is more convenient to perform the Fourier transform of $\phi^{(0)}(u, v)$ with respect to the single variable u:

$$\begin{aligned}\tilde{\phi}(k, v) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}u\, \mathrm{e}^{-\mathrm{i}ku}\phi^{(0)}(u, v)\\ &= \sqrt{2\pi}\left[-(1 + p\Theta(v))\delta(k) + \lim_{\epsilon\to 0}\frac{p}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right].\end{aligned} \tag{3.10}$$

The ordinary product $\phi^{(0)2}$ is mapped, under the Fourier transform, to a convolution product:

$$\begin{aligned}\phi^{(0)2}(u, v) \to{}& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}q\, \tilde{\phi}(q, v)\tilde{\phi}(k - q, v)\\ ={}& \sqrt{2\pi}\left[(1 + p(p+2)\Theta(v))\delta(k) + \lim_{\epsilon\to 0}\frac{p(p-2) - 2p^2\Theta(v)}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right]\end{aligned} \tag{3.11}$$

which is exactly the Fourier transform of

$$\phi^{(0)2}(u, v) = 1 + p(p+2)\Theta(v) + (p(p-2) - 2p^2\Theta(v))\Theta(u). \tag{3.12}$$

The tool that we need during the calculation of the noncommutative case is how the star product of two functions of u and v is translated into a convolution product of their Fourier transforms. It is not difficult to show that $\phi_1 * \phi_2$, defined the Fourier transforms

$$\begin{aligned}\phi_1(u_1, v_1) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}q_1\, \mathrm{e}^{\mathrm{i}q_1u_1}\tilde{\phi}_1(q_1, v_1)\\ \phi_2(u_2, v_2) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}q_2\, \mathrm{e}^{\mathrm{i}q_2u_2}\tilde{\phi}_2(q_2, v_2),\end{aligned} \tag{3.13}$$

is mapped to the following NC convolution product:

$$\widetilde{(\phi * \phi)}(k, v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{d}q\, \tilde{\phi}_1\left(q, v + \frac{\theta}{2}(k - q)\right)\tilde{\phi}_2\left(k - q, v - \frac{\theta}{2}q\right). \tag{3.14}$$

We note from this definition that if the two functions are equal ($\phi_1 = \phi_2$), then, due to the symmetry of the convolution product $q \to k - q$, the dependence on the θ variable becomes even, as happens for the star product of the same function.

Let us calculate the star product $\phi^{(0)} * \phi^{(0)}$ using the rule (3.14)

$$(\widetilde{\phi^{(0)} * \phi^{(0)}})(k, v) = \sqrt{2\pi}\left[(1 + p(p+2)\Theta(v))\delta(k) + \lim_{\epsilon\to 0} \frac{p(p-2) - p^2\big(\Theta\big(v + \frac{\theta}{2}k\big) + \Theta\big(v - \frac{\theta}{2}k\big)\big)}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right]. \tag{3.15}$$

Therefore, the pure noncommutative contribution is

$$(\widetilde{\phi^{(0)} * \phi^{(0)}})_{\mathrm{NC}}(k, v) = -\frac{p^2}{\sqrt{2\pi}\,\mathrm{i}} \lim_{\epsilon\to 0}\left(\frac{\Theta\big(v + \frac{\theta}{2}k\big) + \Theta\big(v - \frac{\theta}{2}k\big) - 2\Theta(v)}{k - \mathrm{i}\epsilon}\right). \tag{3.16}$$

In the following, we will assume for simplicity that $uv > 0$ and $\theta > 0$, otherwise some signs function should be added, with the result of making the notation heavier.

Let us perform the anti-Fourier transform of (3.15) to have a better idea of the noncommutative source:

$$\begin{aligned}(\phi^{(0)} * \phi^{(0)})_{\mathrm{NC}} &= \int_0^\theta \mathrm{d}\theta\, \frac{\partial}{\partial\theta}(\phi^{(0)} * \phi^{(0)})_{\mathrm{NC}} = \frac{1}{\sqrt{2\pi}}\int_0^\theta \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}ku}\frac{\partial}{\partial\theta}(\widetilde{\phi^{(0)} * \phi^{(0)}})_{\mathrm{NC}} \\ &= -\frac{p^2}{\pi}\int_{+\infty}^{\frac{2uv}{\theta}} \mathrm{d}x\, \frac{\sin x}{x} = -\frac{p^2}{\pi}\left[\mathcal{S}\mathrm{i}\left(\frac{2uv}{\theta}\right) - \mathcal{S}\mathrm{i}(+\infty)\right].\end{aligned} \tag{3.17}$$

This function is a special function, known as the sine integral, defined as

$$\mathcal{S}\mathrm{i}(z) = \int_0^z \mathrm{d}x\, \frac{\sin x}{x} \qquad \mathcal{S}\mathrm{i}(+\infty) = \frac{\pi}{2}. \tag{3.18}$$

Therefore, we conclude that the noncommutative part of the source has support only in the small region $0 < uv \lesssim \theta$ (otherwise the contribution is negligible) around the sharp wave packets, and its role is to give a size of order $\sqrt{\theta}$ to the noncommutative kinks.

The next step is to perform the complete calculation of the source (3.9). We will compare again the commutative case with the noncommutative one, to be able to extract the pure noncommutative part. In the commutative case, we need to compute in Fourier transform the product $\phi^{(0)}(\phi^{(0}\phi^{(0)} - 1)$ where

$$\begin{aligned}\tilde{\phi}^{(0)} &= \sqrt{2\pi}\left[-(1 + p\Theta(v))\delta(k) + \lim_{\epsilon\to 0}\frac{p}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right] \\ (\widetilde{\phi^{(0)}\phi^{(0)}} - 1) &= \sqrt{2\pi}\left[p(p+2)\Theta(v)\delta(k) + \lim_{\epsilon\to 0}\frac{p(p-2) - 2p^2\Theta(v)}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right].\end{aligned} \tag{3.19}$$

We obtain

$$\phi^{(0)}(\widetilde{\phi^{(0)}\phi^{(0)}} - 1)(k, v) = \sqrt{2\pi}\left[-p(p+1)(p+2)\Theta(v)\delta(k) + \lim_{\epsilon\to 0}\frac{p(p-1)(p-2) + 6p^2\Theta(v)}{2\pi\mathrm{i}(k - \mathrm{i}\epsilon)}\right]. \tag{3.20}$$

It is easy to verify the correctness of this result, remembering that

$$\phi^{(0)}(\phi^{(0)2} - 1) = -p(p+1)(p+2)\Theta(v) + p(p-1)(p-2)\Theta(u) + 6p^2\Theta(u)\Theta(v). \tag{3.21}$$

Let us compute the noncommutative case:

$$\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1) \tag{3.22}$$

where

$$(\widetilde{\phi^{(0)} * \phi^{(0)}} - 1) = \sqrt{2\pi}\left[p(p+2)\Theta(v)\delta(k) + \frac{1}{2\pi\mathrm{i}}\lim_{\epsilon\to 0}\frac{p(p-2)}{k-\mathrm{i}\epsilon}\right.$$
$$\left. - \lim_{\epsilon\to 0}\frac{p^2\big(\Theta\big(v+\frac{\Theta}{2}k\big)+\Theta\big(v-\frac{\Theta}{2}k\big)\big)}{2\pi\mathrm{i}(k-\mathrm{i}\epsilon)}\right]. \tag{3.23}$$

After simple but tedious calculations we arrive at the following result:

$$\phi^{(0)} * (\widetilde{\phi^{(0)} * \phi^{(0)}} - 1)(k,v) = \sqrt{2\pi}\left[-p(p+1)(p+2)\Theta(v)\delta(k) + \lim_{\epsilon\to 0}\frac{p(p-1)(p-2)}{2\pi\mathrm{i}(k-\mathrm{i}\epsilon)}\right.$$
$$+ \lim_{\epsilon\to 0}\frac{p^3\big(\Theta\big(v+\frac{\theta}{2}k\big)\Theta\big(v-\frac{\theta}{2}k\big)-\Theta(v)\big)}{2\pi\mathrm{i}(k-\mathrm{i}\epsilon)}$$
$$+ \lim_{\epsilon\to 0}\frac{3p^2\big(\Theta\big(v+\frac{\theta}{2}k\big)+\Theta\big(v-\frac{\theta}{2}k\big)\big)}{2\pi\mathrm{i}(k-\mathrm{i}\epsilon)}$$
$$\left. - \frac{p^3}{2(2\pi\mathrm{i})^2}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\mathrm{d}q\,\frac{\Theta(v+\theta q)+\Theta(v-\theta q)-2\Theta(v)}{\big(q+\frac{k}{2}-\mathrm{i}\epsilon\big)\big(\frac{k}{2}-q-\mathrm{i}\epsilon\big)}\right]. \tag{3.24}$$

Let us extract the pure noncommutative part which is made of three parts:

$$\phi^{(0)} * (\widetilde{\phi^{(0)} * \phi^{(0)}} - 1)_{\mathrm{NC}}(k,v,\theta) = I_1(k,v,\theta) + I_2(k,v,\theta) + I_3(k,v,\theta). \tag{3.25}$$

where

$$\begin{aligned}
I_1(k,v,\theta) &= \lim_{\epsilon\to 0}\frac{3p^2\big(\Theta\big(v+\frac{\theta}{2}k\big)+\Theta\big(v-\frac{\theta}{2}k\big)-2\Theta(v)\big)}{\sqrt{2\pi}\mathrm{i}(k-\mathrm{i}\epsilon)}\\
I_2(k,v,\theta) &= \lim_{\epsilon\to 0}\frac{p^3\big(\Theta\big(v+\frac{\theta}{2}k\big)\Theta\big(v-\frac{\theta}{2}k\big)-\Theta(v)\big)}{\sqrt{2\pi}\mathrm{i}(k-\mathrm{i}\epsilon)}\\
I_3(k,v,\theta) &= -\frac{p^3\sqrt{2\pi}}{2(2\pi\mathrm{i})^2}\lim_{\epsilon\to 0}\int_{-\infty}^{+\infty}\mathrm{d}q\,\frac{\Theta(v+\theta q)+\Theta(v-\theta q)-2\Theta(v)}{\big(q+\frac{k}{2}-\mathrm{i}\epsilon\big)\big(\frac{k}{2}-q-\mathrm{i}\epsilon\big)}.
\end{aligned} \tag{3.26}$$

Let us compute the anti-Fourier transforms of I_i:

$$I_i(u,v) = \frac{1}{\sqrt{2\pi}}\int_0^{\theta}\mathrm{d}\theta\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}ku}\frac{\partial}{\partial\theta}I_i(k,v,\theta). \tag{3.27}$$

With the trick of integrating and deriving with respect to θ, all the integrals can be solved, obtaining the following results:

$$\phi^{(0)} * (\phi^{(0)} * \phi^{(0)} - 1)_{\mathrm{NC}}(u,v,\theta) = \frac{3p^2 + p^3(\Theta(v)-\Theta(u))}{\pi}\left[\mathcal{S}\mathrm{i}\left(\frac{2uv}{\theta}\right) - \mathcal{S}\mathrm{i}(+\infty)\right]. \tag{3.28}$$

Let us start integrating this source by taking the position

$$\phi_i^{(1)}(u,v) = \frac{3p^2 + p^3(\Theta(v)-\Theta(u))}{2\pi}\int_0^{\theta}\mathrm{d}\theta\,\phi^{(1)}\left(\frac{2uv}{\theta}\right) \tag{3.29}$$

where

$$\partial_u\partial_v\phi^{(1)}\left(\frac{2uv}{\theta}\right) = -\frac{2\lambda}{\theta}\sin\left(\frac{2uv}{\theta}\right). \tag{3.30}$$

Let us define $x = 2uv/\theta$, then equation (3.30) is equivalent to

$$\partial_x(x\partial_x\phi^{(1)}(x)) = -\lambda \sin x \quad \rightarrow \quad \phi^{(1)}(x) = \lambda\mathcal{C}\mathrm{i}\left(\frac{2uv}{\theta}\right) \tag{3.31}$$

where we have introduced the cosine integral function defined as

$$\mathcal{C}\mathrm{i}(z) = \int_{+\infty}^{z} \mathrm{d}x \, \frac{\cos x}{x}. \tag{3.32}$$

However, the steps functions give rise to additional contributions that we need to subtract

$$\phi_{ii}^{(1)}(u, v) = -\lambda\theta\left(\frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{2\pi}\right)\log\left(\frac{2uv}{\theta}\right). \tag{3.33}$$

We have arrived at the final formula:

$$\begin{aligned}\phi^{(1)} = \lambda\left(\frac{3p^2 + p^3(\Theta(v) - \Theta(u))}{2\pi}\right)\Bigg[\theta\left(\cos\left(\frac{2uv}{\theta}\right) + \mathcal{C}\mathrm{i}\left(\frac{2uv}{\theta}\right) - \log\left(\frac{2uv}{\theta}\right)\right) \\ + 2uv\left(\mathcal{S}\mathrm{i}\left(\frac{2uv}{\theta}\right) - \mathcal{S}\mathrm{i}(+\infty)\right)\Bigg].\end{aligned} \tag{3.34}$$

Fortunately, the divergent terms around the wave packets $uv \sim 0$ cancels out. Again this field has support primarily only in a small region around the wave packets $0 < uv \lesssim \theta$, apart from the oscillating cosine term and a logarithmic term which is less divergent than the corresponding classical term for uv large, as in equation (3.8). The asymptotic states are modified by noncommutativity when the two wave packets scatter each other, while they are not if the shock waves move in the same direction.

4. Conclusions

In this paper, we have discussed an example of scattering which hopefully can be solved at a nonperturbative level. We have compared the classical scattering of kinks in the (1 + 1) Higgs model with the noncommutative case. At a classical level we have found a smooth solution without divergences. This solution is based on introducing an ansatz, which reduces the equations of motions to a single nonlinear differential equation. We have been able to have full control of it by combining analytic and numerical methods. The solution of this equation is similar to a Bessel function of order zero, which contains damped oscillations towards a constant asymptotic value.

In the noncommutative case, the solution we have found is only perturbative; at this level there appear logarithmic terms which are divergent both near the wavefronts and at infinity. However, the whole combination of terms cooperates to eliminate the divergences near the wavefronts. Instead the divergence at infinity cannot be eliminated at a fixed order of perturbation theory, but only re-summing all orders of perturbation theory.

The peculiar characteristic of noncommutativity is to dress the sharp wavefronts of the kinks giving them a size of order $\sqrt{\theta}$ permanently, also when the wavefronts are far apart. Noncommutativity modifies the asymptotic states for two-body scattering, when the two shock waves scatter each other, while it does not alter the classical kink configuration if the two shock waves proceed in the same direction. May be this property complicates the quantization of such theories in terms of the S-matrix approach.

This work leaves open many questions; firstly, it would be nice to solve this model exactly at all orders and prove that there is a smooth solution describing the scattering of noncommutative kinks, re-summing in a finite term the potentially dangerous logarithmic term we have found perturbatively. Then investigating deeply the characteristics of the noncommutative scattering we can look for the right axioms on which to base the quantization of such theories.

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